REMARKS ON CONSERVATIVE MARKOV PROCESSES*

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ABSTRACT

Necessary and sufficient conditions, for a Markov process to be conservative, are studied. As a consequence it is proved: 1. If P is conservative so is P^k. 2. P is conservative if and only if the probability of returning to A, from a point in A, is one.

Let (X,Σ,m) be a σ finite measure space and P an operator on $L_1(X,\Sigma,m)$ with $\| uP \| \leqq \| u \|$, $u \geqq 0 \Rightarrow uP \geqq 0$. Denote the adjoint operator on L_∞ by $Pf: \langle uP,f\rangle = \langle u,Pf\rangle$ then $f \geqq 0 \Rightarrow Pf \geqq 0$ and $P1 \leqq 1$. Let $X\, C \cup D$ be the Hopf decomposition of X into conservative and dissipative parts [see 3, (2.2) and (2.3)].

THEOREM.. *There exists a sequence of functions h_k with the following properties*:

(a) $0 \leqq h_k \leqq h_{k+1} \leqq 1$.

(b) $h_k(x) = 0$ *if* $x \in C$, $h_k(x) > 0$ *if* $x \in D$.

(c) $Ph_k \leqq h_k$ *and* $\lim_{n\to\infty} P^n h_k = 0$.

(d) *If* $D_k = \{x : h_k(x) = 1\}$ *then* $D_k \uparrow D$.

Proof. Let $0 < u \in L_1$ then $\sum_{n=0}^{\infty} uP^n(x) < \infty$ for $x \in D$. Choose $f \in L_\infty$ such that $f(x) = 0$ $x \in C$, $f(x) > 0$ $x \in D$ and $\langle \sum_{n=0}^{\infty} uP^n, f\rangle < \infty$. By Fatou's Lemma $\langle u, \sum_{n=0}^{\infty} P^n f\rangle \infty <$ and since $u > 0$ it follows that $g = \sum_{n=0}^{\infty} P^n f < \infty$. Now note that $g(x) = 0$ if $x \in C$ by [3, (2.4)]. Also

$$g(x) > 0 \; x \in D, \; Pg \leqq g \text{ and } P^j g = \sum_{n=j}^{\infty} P^n f_{j\to\infty} \to 0.$$

We apply P to g, which is not in L_∞ by [3, (1.9)]. Put $h = \min(g,1)$ then:

$0 \leqq h \leqq 1$, $0 < h(x)$ $x \in D$, $h(x) = 0$ $x \in C$, $Ph \leqq h$ and $P^j h \to 0; j \to \infty$:
$Ph \leqq \min(Pg, P1) \leqq \min(g,1) = h$ and $P^j h \leqq P^j g \to 0$.

Finally let us define inductively $h_1 = h$ $h_k = k \min((1/k), h_{k-1})$. If $h_{k-1}(x) \geqq (1/k)$ then $h_k(x) = 1 \geqq h_{k-1}(x)$ but if $h_{k-1}(x) < (1/k)$ $h_k(x) = kh_{k-1}(x) \geqq h_{k-1}(x)$ which proves (a). Condition (b) is clear. Now if (c) holds for h_{k-1} then

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$$Ph_k \leqq k \min \left(P \frac{1}{k}, Ph_{k-1}\right) \leqq k \min \left(\frac{1}{k}, h_{k-1}\right) = h_k$$

and $P^n h_k \leqq kP^n h_{k-1} \to 0$.

Finally if $x \in D$ then $h(x) = h_1(x) > (1/k)$ for some k so $h_{k-1}(x) > (1/k)$ and $h_k(x) = 1$.

Corollary. 1. *The following conditions are equivalent*:

1. $X = C$.
2. *If* $0 \leqq f < \infty$ *and* $Pf \leqq f$ *then* $Pf = f$.
3. *If* $0 \leqq f \leqq 1$ *and* $Pf \leqq f$ *then* $Pf = f$.
4. *If* $0 \leqq f \leqq 1$ *and* $Pf \leqq f$ *and* $\lim_{n\to\infty} P^n f = 0$ *then* $f = 0$.
5. There is no function f with $0 \leqq f \leqq 1$, $Pf \leqq f$, $\lim_{n\to\infty} P^n f = 0$ *and* $m\{x : f(x) = 1\} \neq 0$.

Proof. $1 \Rightarrow 2$ by [3, Chapter II, Theorem B]. $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ is obvious. Finally, if 1 is false so is 5 by the Theorem.

Remark. If P is induced by a measurable transformation and $X \neq C$ then the function f of 5 can be chosen to be a characteristic function [see 3, (2.6)]. If (X, Σ, m, P) is the one-dimensional discrcete random walk with $p \neq 1/2$ [see 1, p. 23] then $X = D$ but if $P1_A \leqq 1_A$ then either $A = X$ or $A = \emptyset$.

Corollary 2. *If P is conservative so is P^k.*

Proof. Use 3 of Corollary 1: If $0 \leqq f \leqq 1$ and $P^k f \leqq f$ then

$$(I - P)((I + P + \cdots + P^{k-1})f) \geqq 0 \text{ and thus}$$

$$0 = (I - P)((I + P + \cdots + P^{k-1})f) = (I - P^k)f.$$

Remark. We followed here the proof of [4] for processes induced by measurable transformations.

For the last Corollary let us introduce some notation:

Let $A \in \Sigma$ $m(A) > 0$ put

$$i_A = \sum_{k=0}^{\infty} (T_{A'} P)^k 1_A$$

where $(T_B f)(x) = 1_B(x) f(x)$. Then i_A is the smallest function that satisfies

$$1_A \leqq i_A \leqq 1, \; Pi_A \leqq i_A. \text{ [See 3, (3.2)].}$$

Also

$$Pi_A = \lim_{N\to\infty} \sum_{k=0}^{N} (PT_{A'})^k P1_A = \lim_{N\to\infty} \sum_{n=0}^{N} (PT_{A'})^k P(1 - T_{A'} 1)$$

$$\leqq \lim_{N\to\infty} \left[\sum_{n=0}^{N} (PT_{A'})^k 1 - \sum_{n=0}^{N} (PT_{A'})^{k+1} 1\right] = 1 - \lim_{N\to\infty} (PT_{A'})^{N+1} 1.$$

Now the sequence $(T_{A'}PT_{A'})^n 1$ is monotone and bounded let its limit be g. Then

$$P(T_{A'}PT_{A'})^n 1 = (PT_{A'})^{n+1} 1 \to Pg$$

and $T_{A'}PT_{A'}g = g$ hence $Pg \geqq g$ and $T_{A'}g = g$.

Thus *if* $A \subset C$ $Pg(x) = g(x)$ $x \in A$ by [3, (2.9)], hence if

$$A \subset C \lim (PT_{A'})^{n+1} 1(x) = 0 \; x \in A \text{ and so } Pi_A(x) = 1 \; x \in A.$$

COROLLARY 3. *$X = C$ if and only if for every $A, m(A) > 0$, $Pi_A(x) = 1$ $x \in A$.*

Proof If $X = C$ we proved the condition above. If D is not empty take the function f as in 5, Corollary 1 and let E be any set with $0 \neq 1_E \leqq f$. Now $PPi_E \leqq Pi_E$ and if $Pi_E \geqq 1_E$ then it follows from the minimality of i_E that $Pi_E \geqq i_E$ but then $1_E \leqq i_E \leqq \lim_n P^n i_E \leqq \lim_n P^n f = 0$ which contradicts 5.

REMARK. If one uses the sequence D_k of (*d*) of the Theorem it follows that for any set $E \subset D_k$ $m(\{x : Pi_E(x) \neq 1\} \cap E) \neq 0$.

Corollary 3 is proved in [2, Theorem 1.1] by different methods.

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